Y01

Proceedings of the 5th International Conference on Inverse problems in Engineering: Theory and Practice, Cambridge, UK,11-15th July, 2005

SOLVING INVERSE COUPLE-STRESS PROBLEMS VIA AN AGGREGATE FUNCTION APPROACH

YANG HAITIAN, CHEN GUOSHENG, ZHENG CHANGLIANG, XUE QIWEN, ZHANG ZHIFENG
State Key Lab. of Structural Analysis for Industrial Equipment,
Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, P. R. China
e-mail: Haitian@dlut.edu.cn

Abstract-This paper presents a general numerical model to solve the inverse couple-stress problem that is formulated as a nonlinear programming problem with multi-constraints of inequality. By utilizing an aggregate function approach, multi-constraints are converted into a single smooth constraint, resulting in computing convenience. A newly developed discrete Cosserat finite element approach is employed in modeling the direct problem, and the Brayden-Fletcher-Goldfarb-Shanno (BFGS) algorithm is combined with a technique of multiplier penalty functions in the process of solving the inverse problem. Numerical verification is given with the consideration of noisy data.

1. INTRODUCTION

Couple stress theory can be traced back to 1887 when Voigt assumed the existence of couple stress. In 1909, the Cosserat brothers first set up a framework of couple stress theory which has been further developed since then, [8,9,14,15]. The couple stress theory assumes that the interaction of the material on two sides of a surface element is equipollent to a force and a couple (couple stress). Accordingly, a group of variables including moments, curvatures, and characteristic length are introduced within a continuum framework, [2].

One important aspect of the application of couple stress theory was to describe the properties of microstructure of materials, such as materials with granular, fibrous and lattice structures, [14]. For some cases where the size effects have to be taken into account, [4], this theory was employed to explain the variation of hardening behavior, [11], and to soften local singularities, [5].

The study of this paper is motivated by the question that if a continuum couple stress model is adopted, how to determine its constitutive coefficients, including the so-called characteristic length l?

Determining these coefficients is one of the key issues of inverse couple stress problems. However, to the best of the authors' present knowledge it seems no reports exist directly related to this matter. Thus this paper proposes a nonlinear programming model with multi-constraints to solve an inverse couple-stress problem with unknown constitutive coefficients. A kind of newly developed discrete Cosserat finite element method (FEM), [20], is employed in the solution of direct problems. By exploiting a maximum entropy theory based aggregate function method, [12], multi-constraints can be converted into a single differentiable constraint without distinguishing active and inactive constraints in the iterative process. The optimization with a single constraint is realized using a technique of multiplier penalty functions. Satisfactory results are shown in the numerical verification, and the effects of noisy data on the results are taken into account.

2. GOVERNING EQUATIONS FOR DIRECT COUPLE STRESS PROBLEMS

For plane couple-stress problems in the absence of body forces and couples, the equilibrium equations are given by, [2],

$$\begin{cases} \sigma_{x,x} + \tau_{yx,y} = 0 \\ \tau_{xy,x} + \sigma_{y,y} = 0 \\ \mu_{x,x} + \mu_{y,y} + \tau_{xy} - \tau_{yx} = 0 \end{cases} \qquad \boldsymbol{x} \in \Omega \tag{1}$$

where $\boldsymbol{\sigma}^T = \{\sigma_x, \sigma_y, \tau_{xy}, \mu_x, \mu_y\}^T$ refers to the stress vector, $\{\sigma_x, \sigma_y, \tau_{xy}\}$ are the Cauchy components of the stress vector, and μ_x and μ_y denote the moments.

The relationship of strain and displacement is described by, [2],

$$\boldsymbol{\varepsilon} = \boldsymbol{L}\boldsymbol{u} \tag{2}$$

where $\boldsymbol{\varepsilon}^T = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \kappa_x, \kappa_y\}^T$ represents the strain vector, $\boldsymbol{u} = \{u, v, \theta_z\}^T$ designates the vector of displacement, and θ_z is a microrotation about the *z* axis defined by

$$\theta_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{3}$$

$\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are the Cauchy components of the strain vector, κ_x and κ_y designate the curvatures corresponding to μ_x and μ_y , and they are specified by

$$\begin{aligned} \kappa_x &= \theta_{z,x} \\ \kappa_y &= \theta_{z,y} \end{aligned} \tag{4,5}$$

$$\boldsymbol{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \tag{6}$$

The boundary conditions are specified by, [4],

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \qquad \boldsymbol{x} \in \Gamma_u \tag{7}$$

$$\begin{cases} \sigma_{ij} n_j = T_i^0 \\ \mu_j n_j = q^0 \end{cases} \qquad \boldsymbol{x} \in \Gamma_\sigma \tag{8,9}$$

where σ_{ij} is the component of Cauchy stress and μ_j is the component of couple stress. The vector $\overline{\boldsymbol{u}}$ contains the prescribed values of $\boldsymbol{u}$ on Γ_u . Furthermore T_i^0 and q^0 are the prescribed vectors of traction and moment on Γ_σ , where n_j denotes the unit outward normal to the boundary, $\Gamma_u + \Gamma_\sigma = \Gamma = \partial\Omega$, $\boldsymbol{x}$ represents a vector of coordinates and the subscripts *u* and σ of Γ refer to displacement and stress, respectively.

3. IMPLEMENTATION OF FEM ON DIRECT COUPLE STRESS PROBLEMS

Since the analytical solution is usually difficult to obtain, a number of numerical approaches have been developed for the solution of eqns (1)-(8), [1,16]. In this paper a kind of newly developed discrete Cosserat triangular finite element approach, [20], is employed to solve direct couple stress problems.

Within a triangular finite element, as shown in the Figure 1, the vector $\boldsymbol{u} = \{u, v, \theta_z\}^T$ is interpolated by

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \tag{10}$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 \tag{11}$$

$$\theta_z = N_1\theta_1 + N_2\theta_2 + N_3\theta_3 + aN_q \tag{12}$$

where $N_1 = 1 - \xi - \eta$, $N_2 = \xi$, $N_3 = \eta$, $N_q = 27\xi\eta(1 - \xi - \eta)$, N_q is a bubble function adopted for improving accuracy, a is an inner parameter which can be determined by imposing $\theta_z = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ at the centroid of an element, and (ξ, η) are coordinates defined in Figure 2.

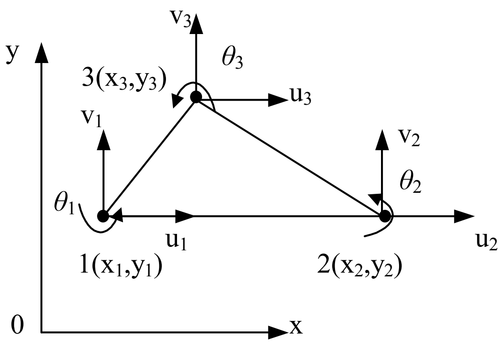

Figure 1. A triangular element.

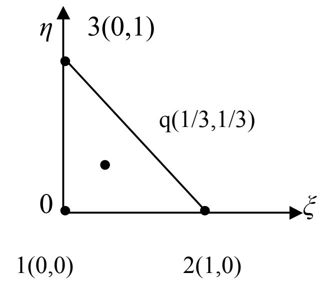

Figure 2. A triangular element in the natural coordinate system.

Equations (10)-(12) can further be written as

$$\boldsymbol{u} = [N]\,\boldsymbol{d}^e \tag{13}$$

where $[N]$ designates a matrix of shape functions.

$$\boldsymbol{d}^e = \{u_1, v_1, \theta_{z1}, u_2, v_2, \theta_{z2}, u_3, v_3, \theta_{z3}\}^T \tag{14}$$

The stiffness matrix of an element can be expressed by

$$\boldsymbol{K}^e = \iint_{\Omega_e} [B]^T [D][B] F dxdy = \iint_{\Omega_e} [B]^T \left(b_1[H_1] + b_2[H_2] + b_3[H_3]\right)[B] F dxdy \tag{15}$$

where F represents the thickness of the material, $[D]$ is a constant matrix related to the elastic modulus E, Poisson's ratio v, and the characteristic length l of the material, in addition $[B] = \boldsymbol{L}[N]$.

For the plane stress problem

$$[D] = \begin{bmatrix} \frac{E}{1-v^2} & \frac{Ev}{1-v^2} & & & \\ \frac{Ev}{1-v^2} & \frac{E}{1-v^2} & & & \\ & & \frac{E}{2(1+v)} & & \\ & & & 4\beta & \\ & & & & 4\beta \end{bmatrix} \tag{16}$$

where $\beta = l^2 G = l^2 \dfrac{E}{2(1+v)}$ is called the curvature modulus, [2].

$[D]$ can further be divided into $[D] = b_1[H_1] + b_2[H_2] + b_3[H_3]$ (17)

where $b_1 = \frac{Ev}{1-v^2}, b_2 = \frac{E}{1+v},\ b_3 = 4\beta$

$$[H_1] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, [H_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, [H_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

By utilizing virtual displacement principle and assembling all elements over the domain, a system of equations can be obtained, [20], having the form

$$\boldsymbol{K}\ \tilde{\boldsymbol{u}} = \boldsymbol{F}_1 \tag{19}$$

where $\boldsymbol{K} = \sum \iint_{\Omega_e} [B]^T \left(b_1[H_1] + b_2[H_2] + b_3[H_3]\right)[B]Fdxdy$ denotes the stiffness matrix of the system

and $\tilde{\boldsymbol{u}}$ and $\boldsymbol{F}_1$ refer to general nodal vectors of displacement and force, respectively.

4. INVERSE COUPLE-STRESS PROBLEM

For the inverse problem given by eqn.(19), $\tilde{\boldsymbol{u}}$ is partially known. The unknowns to be determined are $\boldsymbol{x} = \{E, v, l\}$ or $\boldsymbol{b} = \{b_1, b_2, b_3\}$ in the stiffness matrix $\boldsymbol{K}$ on the left hand side of eqn.(19).

$\boldsymbol{x}$ or $\boldsymbol{b}$ can be evaluated by minimizing a functional defined by

$$\Pi = \frac{1}{2}(\boldsymbol{H}\tilde{\boldsymbol{u}} - \boldsymbol{u}^*)^T(\boldsymbol{H}\tilde{\boldsymbol{u}} - \boldsymbol{u}^*) = \frac{1}{2}\boldsymbol{R}^T\boldsymbol{R} \tag{20}$$

where $\boldsymbol{u}^*$ denotes a vector of known quasi-static displacements which is usually obtained by measurement; $\tilde{\boldsymbol{u}}$ is given by eqn.(19), $\boldsymbol{H}$ is a transformation matrix mapping the relationship of location between $\tilde{\boldsymbol{u}}$ and $\boldsymbol{u}^*$.

The constraints can be described by

$$b_i > 0 \qquad i = 1,2,\ldots,m \tag{21}$$

Equation (21) represents physical requirements for the constitutive parameters. Without this constraint, numerical oscillation, lower convergence rates, and even divergence may occur in the iterative process of unconstrained optimization, especially in the case which does not guarantee the convexity of the proposed problems, [19].

The sensitivity of Π with respect to $\boldsymbol{b}$ is given by

$$\frac{\partial \Pi}{\partial b_j} = \boldsymbol{R}^T \boldsymbol{H} \frac{\partial \tilde{\boldsymbol{u}}}{\partial b_j} \tag{22}$$

where

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial b_j} = -\boldsymbol{K}^{-1}\frac{\partial \boldsymbol{K}}{\partial b_j}\tilde{\boldsymbol{u}} = -\boldsymbol{K}^{-1}\left(\sum \iint_{\Omega_e} [\boldsymbol{B}]^T [H_j][\boldsymbol{B}]\boldsymbol{F}dxdy\right)\tilde{\boldsymbol{u}} \tag{23}$$

5. IMPLEMENTATION OF AGGREGATE FUNCTION METHOD

Multi-constraints defined by eqn.(21) can be converted into a single differentiable constraint via a maximum entropy theory based on aggregate function method, [12,13], the trouble caused by distinguishing active and inactive constraints in the iterative process can therefore be avoided. Furthermore, some well developed algorithms, such as quasi-exact penalty function algorithm, multiplier penalty functions algorithm etc., can be exploited, [12,13].

Consider a problem defined by

$$\text{(P)} \quad \begin{cases} \min & f(\boldsymbol{Y}) \\ Subject\ \ to & g_i(\boldsymbol{Y}) \le 0, \quad i = 1,2,\cdots,m \end{cases} \quad \boldsymbol{Y} \in R^n \tag{24}$$

where $\boldsymbol{Y}$ is a vector of variables, $f(\boldsymbol{Y})$ and $g_i(\boldsymbol{Y})$ are smooth nonlinear functions of $\boldsymbol{Y}$.

The problem (P) can be converted into an equivalent problem with a single constraint

$$\text{(P1)} \quad \begin{cases} \min & f(\boldsymbol{Y}) \\ Subject\ \ to & \gamma(\boldsymbol{Y}) \le 0 \end{cases} \tag{25}$$

where the single constraint is termed as 'maximum' constraint, having the form

$$\gamma(\boldsymbol{Y}) = \max_i \{g_i(\boldsymbol{Y})\} \tag{26}$$

The problems of (P1) and (P) are definitely equivalent since they have the same feasible regions. Due to the non-differentiability of eqn.(26), g_p, a 'surrogate constraint' or 'aggregate function' , was proposed by Li [12,13] to smooth the constraint, namely

$$g_p(\boldsymbol{Y}) = \left(1/p\right) \ln\left\{\sum_{i=1}^{m} \exp[p \cdot g_i(\boldsymbol{Y})]\right\} \tag{27}$$

where p is a positive parameter.

There exists an inequality relationship, [13],

$$\gamma(\boldsymbol{Y}) \le g_p(\boldsymbol{Y}) \le \gamma(\boldsymbol{Y}) + \ln(m)/p \tag{28}$$

where m is the number of constraints.

Li [12,13] proved that $g_p(\boldsymbol{Y})$ will approach $\gamma(\boldsymbol{Y})$ uniformly in R^n when p tends to infinity. Thus the problem (P1) with a non-smooth constraint (25) is equivalent to a problem with a single smooth constraint, i.e.

$$\text{(P2)} \quad \begin{cases} \min \quad f(\boldsymbol{Y}) \\ Subject\ \ to \quad g_p(\boldsymbol{Y}) = \left(1/p\right) \ln\left\{\sum\limits_{i=1}^{m} \exp\left[p \cdot g_i(\boldsymbol{Y})\right]\right\} \le 0 \end{cases} \tag{29}$$

Hence $g_p(\boldsymbol{Y})$ represents an integral effect of all the constraints. The adoption of $g_p(\boldsymbol{Y})$ can make computing more efficient, [13].

When problem (P) has at least one 'active' constraint，the single inequality constraint of (P2) can be further written as an equality constraint, [7,17]. By means of multiplier penalty functions, [17], the problem (P2) can be treated as an unconstrained optimization defined by

$$\text{(P3)} \quad \min \quad \Phi_p(\boldsymbol{Y}, \alpha) = f(\boldsymbol{Y}) + \alpha \cdot g_p(\boldsymbol{Y}) + c \cdot g_p^2(\boldsymbol{Y})/2 \tag{30}$$

where c is a penalty factor and α is a Lagrange multiplier associated with the single constraint (27) and α is equal to the sum of all the Lagrange multipliers in the problem (P), [13].

In the iterative process, α will be updated by

$$\alpha^{k+1} = \alpha^k + c \cdot g_p(\boldsymbol{Y}^k) \tag{31}$$

In order to solve eqn.(30), a standard BFGS algorithm, [7], for unconstrained optimization is employed.

The major steps of solving eqn.(30) via the BFGS algorithm can be referred to [19].

6. NUMERICAL EXAMPLES AND REMARKS

By considering eqns (20) and (21) as Problem (P), and using the techniques proposed in section 4, a number of numerical tests are carried out.

Consider a square plate subjected to a tension $q = 20y/L$ as shown in Figure 3 where L =0.16. $\boldsymbol{u} = \{u, v, \theta_z\}^T$ occurring under such loading condition will contain the information related to all the material parameters. Figure 4 gives the description of the mesh configuration. In the homogeneous case, due to the symmetry of the problem, only a quarter of the plate is computed. In the non-homogeneous case, elements 1-16 and 17-32 are assumed to have different material parameters, respectively.

Noisy data is specified by, [18],

$$\boldsymbol{u}^* = (1 + \delta \cdot \xi) \cdot \boldsymbol{u}^e \tag{32}$$

where $\boldsymbol{u}^*$ represents the known information of quasi-static displacements with noisy data, ξ is a random variable which follows a normal distribution with zero mean and unit standard deviation and δ denotes a deviation.

For each fixed value of δ, 40 groups of results are obtained with 40 ξ produced randomly.

The confidence interval is evaluated by, [18],

$$\bar{x} \pm \frac{t_{(\beta/2,\ N-1)} * s}{\sqrt{N}} \tag{33}$$

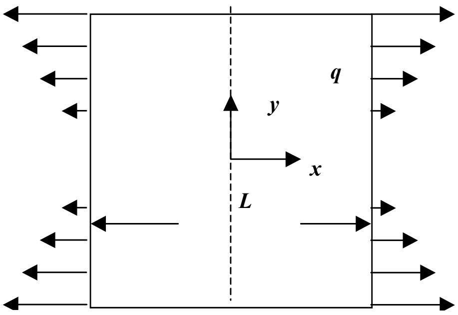

Figure 3. A foursquare plate subjected to a tension.

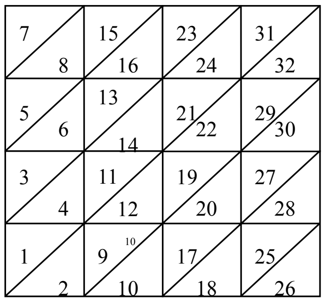

Figure 4. Finite element mesh.

where $\bar{x}$ represents the mean of identified parameters, s is a standard deviation of the identified parameters, t denotes a t distribution with the degree of freedom (N-1), N is the capability of samples, and the confidence level is $1-\beta$.

Tables 1 and 2 give the results without noisy data for homogeneous and inhomogeneous cases, respectively. Table 3 represents the effect of initial guesses on the results. Table 4 exhibits the results with a confidence interval of 95% for noisy data.

Table 1. Identification of constitutive parameters in the homogeneous case.

Constitutive parameters	Initial guesses	Numerical results	Actual values	Number of sample points	Number of iterations
E	1.0000E+06	1.0000E+05	1.0000E+05		
v	4.0000E-01	3.0000E-01	3.0000E-01	25	2
l	2.0000E-04	1.0000E-05	1.0000E-05		

Table 2. Identification of constitutive parameters in the inhomogeneous case.

Constitutive parameters	Initial guesses	Numerical results	Actual values	Number of sample points	Number of iterations
E_1	1.0000E+07	1.0000E+05	1.0000E+05	25	17
v_1	4.0000E-01	3.0000E-01	3.0000E-01		
l_1	4.0000E-04	1.0000E-04	1.0000E-04		
E_2	1.0000E+08	1.0000E+06	1.0000E+06		
v_2	5.0000E-01	3.9998E-01	4.0000E-01		
l_2	2.0000E-04	2.9998E-04	3.0000E-04		

Table 3. The effects of initial guesses on the results (Maximum number of iterations is 17).

Constitutive parameters	**1**		**2**		**3**		Actual values
	Initial guesses	Final values	Initial guesses	Final values	Initial guesses	Final values	
E_1	1.000E+04	1.000E+05	1.000E+06	1.000E+05	1.000E+07	1.000E+05	1.000E+05
v_1	4.000E-01	3.000E-01	4.000E-01	3.000E-01	4.000E-01	3.000E-01	3.000E-01
l_1	7.000E-02	9.980E-03	7.000E-02	9.980E-03	7.000E-02	9.980E-03	1.000E-02
E_2	1.000E+05	1.000E+06	1.000E+07	1.000E+06	1.000E+08	1.000E+06	1.000E+06
v_2	5.000E-01	3.999E-01	5.000E-01	3.999E-01	5.000E-01	3.999E-01	4.000E-01
l_2	2.000E-02	2.999E-02	2.000E-02	2.999E-02	2.000E-01	2.999E-02	3.000E-02

Table 4. The effect of noisy data on the results

Constitutive parameters	δ =0.01		δ =0.03		Actual values
	Expected values	Confidence intervals	Expected values	Confidence intervals	
E_1	1.00000E+05	9.91582E+04 1.00842E+05	9.51120E+04	9.26880E+04 9.75560E+04	1.00E+05
v_1	3.00000E-01	2.99089E-01 3.00912E-01	2.94688E-01	2.92031E-01 2.97344E-01	3.00E-01
l_1	9.98269E-03	9.96487E-03 1.00005E-03	1.00875E-02	1.00351E-02 1.01399E-02	1.00E-02
E_2	1.00000E+06	9.91845E+05 1.00816E+06	9.62582E+05	9.48872E+05 9.76291E+05	1.00E+06
v_2	4.00002E-01	3.98761E-01 4.01242E-01	3.92764E-01	3.89145E-01 3.96383E-01	4.00E-01
l_2	3.29257E-02	3.28039E-02 3.30476E-02	3.36524E-02	3.32891E-02 3.40158E-02	3.00E-02

On the basis of the above numerical tests, some remarks can be given as follows:

(i) The proposed approach is capable of solving inverse couple-stress problems with unknown constitutive parameters including characteristic length.

(ii) For most cases, solutions can be achieved within a few iterations.

(iii) The choice of initial guess shows a slight effect on the final results, however the number of iterations will be affected.

(iv) iii The results of identification seem relatively sensitive to noisy data.

(v) Similar remarks to those given in [19] can be made for the choices of p , c and α , and their effects on the solutions.

7. CONCLUSIONS

The major contribution of this paper is to present a general numerical model solving inverse couple stress problems with unknown constitutive coefficients. Since there seems no report directly related to this issue, the present work may be considered as an imperfect start, and considerable further effort is definitely required. The present work is also a new application of the aggregate function method that is numerically proved to be capable of solving inverse couple stress problems with noisy data. A numerical model, facilitated by the sensitivity analysis of displacement with respect to constitutive parameters, is established to solve direct couple stress problems by utilizing a newly developed discrete Cosserat finite element method. In order to avoid possible numerical oscillation, lower convergence rate, and divergence in the iterative process of solving inverse problems, constraints of lower bounds for constitutive parameters are taken into account, and are converted into a single smooth constraint by virtue of an aggregate function method with a consideration of computing convenience. The BFGS algorithm is combined with a technique of multiplier penalty functions in the solving process, and satisfactory numerical verification with noisy data is given.

Acknowledgement

The research leading to this paper is funded by the overseas returnee initiating fund [1999-363], the key project fund [99149], and backbone faculty fund [2000-65], all of which came from the National Education Department of P. R. China. The research is also funded by NSF (10421002), NSF(10472019), NSF(10172024), NKBRSF [G1999032805], and the fund of discipline leaders of the young and middle age faculty in colleges of Liaoning Province.

REFERENCES

1. T. Adachi, Y. Tomta and M. Tanaka, Computational simulation of deformation behavior of 2d-lattice continuum. *Int. J. Mech. Sci.* (1998) **40**, 857-866.

2. A.P. Boresi, and K. P. Chong. *Elasticity in Engineering Mechanics*, John Wiley & Sons, New York, 2000.

3. R. De Borst, Simulation of strain localization: a reappraisal of the Cosserat continuum. *Eng.Comput.* (1991) **8**, 317-32.

4. C. Chen and N. A. Fleck, Size effects in the constrained deformation of metallic foams. *J.Mech.Phys. Solids,* (2002) **50(**5), 955-977.

5. J.Y. Chen, Y. Wei, Y. Huang, J.W. Hutchinson and K.C. Hwang, The crack tip fields in strain gradient plasticity: The asymptotic and numerical analysis. *Eng. Fract. Mech.* (1999) **64**, 625-648.

6. S.H. Chen and T.C. Wang, A new deformation theory with strain gradient effects. *Int. J. of Plasticity*. (2002) **18**, 971-995.

7. Cheng Gengdong, *Introduction of Optimal Designs of Engineering Structures*, China Water Power Press, Beijing (*in Chinese*), 1984.

8. Cosserat E. and Cosserat F, *Théorie des Corps Déformables, Librairie Scientifique*, A. Hermann et Fils, Paris, 1909.

9. A.C. Eringen, Linear theory of micropolar elasticity. *J. Math. Mech.* (1966) **15**, 909-923.

10. A.C. Eringen, Polar and nonlocal field theories. *In: Continuum Physics,* **4***,* Academic Press, London, 1976.

11. N.A. Fleck, G.M. Muller, M.F. Ashby and J.W. Hutchinson, Strain gradient plasticity: theory and experiment. *Acta Metall. Mater.* (1994) **42**, 475-487.

12. Li Xingsi, An aggregate function method for nonlinear programming. *Science in China* (*Series A*) (1991) **21(**12), 1283-1288.

13. Li Xingsi, An efficient approach to a class of non-smooth optimization problems. *Science in China* (*Series A*) (1994) **24**(4), 371-377.

14. R. D. Mindlin and H. F.Tiersten, Effects of couple stress concentrations. *Exp. Mech.* (1963) **3**, 1-7.

15. R.A. Toupin, Elastic materials with couple stresses. *Arch.Rat.Mech. Anal.* (1962) **11**, 385-414.

16. R.D. Wood, Finite element analysis of plane couple-stress problems using first order stress functions. *Int. J. Num. Meth. Engng.* (1988) **26**, 489-509.

17. Tang Huanwen and Qin Xuezhi, *Practical Methods of Optimization*, Dalian University of Technology Press, Dalian (in Chinese), 2000.

18. Wang Denggang, Liu Yingxi and Li Shouju, Regularization procedure for two-dimensional steady heat conduction inverse problems. *Chinese J. Acta Scientiarum Naturalium, Universitatis Jilinensis* (2000) **2**, 55-60.

19. Yang Haitian and Yan Jun, Aggregate function approach based solutions for inverse viscoelasticity problems. *Int. J. Solids and Structures* (2003) **40**, 3707-3716.

20. Zheng Changliang, Ren Mingfa, Zhang Zhifeng and Song Heping, Analysis of Cosserat medium via discrete Cosserat finite element method. *Chinese J. of Comp. Phy.* (2005), accepted.